Several results on robot manipulator motion control require a uniform bound for the Hessian of the potential energy or equivalently the Jacobian of the gravity vector. Not all robot manipulators, however, ensure the existence of such a uniform bound. The first contribution of this article is the complete characterization of this class which is referred to as class $BBGG$ manipulators. The uniform bound of the Hessian is typically part of the control law expression and hence it plays an important role in controller gain synthesis. The second contribution of this article consists of deriving, for class $BBGG$ robot manipulators, an easy to compute explicit expression of the uniform bound in terms of kinematic and inertial link parameters. If for a particular robot manipulator the Hessian of potential energy is not uniformly bounded, a bound exists that is valid within the physical workspace of the manipulator. The third contribution of this article is the derivation of an explicit expression for the latter bound which is useful in the design and controller gain synthesis of control laws that are valid locally.

1. INTRODUCTION

The area of motion control of robot manipulators received a considerable amount of attention over the past two decades resulting in a rich body of control results. The success of these results are primarily due to the exploitation of some very important properties of the robot equations of motion. The standard model for the dynamics of an $n$-degree-of-freedom ($n$-DOF) rigid robot manipulator is as follows:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$ (1)

where the $n$-vectors $q$, $\dot{q}$, and $\ddot{q}$ represent the link variables, velocities, and accelerations, respectively.
Among the properties that are frequently exploited in control, the skew symmetry of the matrix \( [D - 2C(q, \dot{q})] \) has received much attention. Another well known property is the linearity in the parameters of the equations of motion which is exploited in adaptive control. A very important property that has received relatively less attention, though widely assumed, is the uniform boundedness of some of the terms arising in the equations of motion (1).

We refer to a quantity \( f(q) \) as being uniformly bounded when there exists a constant \( c \), independent of \( q \), such that \(|f(q)| \leq c \) for all \( q \in \mathbb{R}^n \). This concept can be extended to a matrix \( M(q) \) by requiring the existence of a constant \( c \) such that \( \|M(q)\| \leq c \) for all \( q \in \mathbb{R}^n \). Among the terms in (1) that are typically assumed to be uniformly bounded is the inertia matrix \( D(q) \), the \( C(q, \dot{q}) \) matrix, and the Jacobian of the gravity vector, \( \partial g(q)/\partial q \), or equivalently the Hessian of the potential energy. That is, in the stability analysis of many control laws, it is assumed that uniform bounds \( \sigma_1, \sigma_2, \gamma, \) and \( \beta \) exist such that

\[
\|D(q)\|_{\text{min}} > \delta_1, \quad \|D(q)\|_{\text{max}} < \sigma_2
\]

\[
\|C(q, \dot{q})\| \leq \gamma \|\dot{q}\|
\]

and

\[
\left\| \frac{\partial g(q)}{\partial q} \right\| \leq \beta \quad \forall q \in \mathbb{R}^n
\]

where \( \|D(q)\|_{\text{min}} \) and \( \|D(q)\|_{\text{max}} \) are, respectively, the minimum and maximum eigenvalues of \( D(q) \).

It is well known that these terms are uniformly bounded for \( \mathcal{RP} \cdots \mathcal{RP} \) robot manipulators (i.e., robots with joints that are all revolute). They are, however, not uniformly bounded for any general robot manipulator. For example, consider the revolute-prismatic (\( \mathcal{RP} \)) robot manipulator of Figure 1 for which the terms in question are given by

\[
D(q) = \begin{bmatrix} d_{1,1} & 0 \\ 0 & m_2 \end{bmatrix}
\]

\[
C(q, \dot{q}) = \begin{bmatrix} \dot{q}_2 m_2 (l_c + q_2) & \dot{q}_1 m_2 (l_c + q_2) \\ -\dot{q}_1 m_2 (l_c + q_2) & 0 \end{bmatrix}
\]

\[
\frac{\partial g(q)}{\partial q} = \begin{bmatrix} -m_1 l_1 g \sin(q_1) - m_2 (q_2 + l_c) g \sin(q_1) & m_2 g \cos q_1 \\ m_2 g \cos q_1 & 0 \end{bmatrix}
\]

where \( d_{1,1} = m_1 l_1^2 + m_2 (q_2 + l_c)^2 + I_1 + I_2 \). It can be seen that \( d_{1,1} \) can be made arbitrarily large by selecting \( q_2 \) sufficiently large. Therefore, a constant \( \sigma_2 \) (independent of \( q \)) satisfying (2) does not exist for this robot manipulator. Thus the inertia matrix is not uniformly bounded. It can be seen that the \( C(q, \dot{q}) \) matrix and the Hessian of potential energy are also not uniformly bounded for this robot manipulator for the same reason.

At this point a reader might question as to why the uniform boundedness is important since all of the terms in question are bounded within the workspace of the robot manipulator. That is, for any given robot manipulator it is always possible to find constant bounds, \( \sigma_1', \sigma_2', \gamma' \), and \( \beta' \) such that

\[
\|D(q)\|_{\text{min}} > \delta_1', \quad \|D(q)\|_{\text{max}} < \sigma_2'
\]

\[
\|C(q, \dot{q})\| \leq \gamma' \|\dot{q}\|
\]

and

\[
\left\| \frac{\partial g(q)}{\partial q} \right\| \leq \beta' \quad \forall q \in W
\]
Figure 1. A revolute-prismatic (2 DOF) robot manipulator.

where \( W \) is the workspace of the robot manipulator defined as follows:

\[
W = \left\{ \begin{array}{c} \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{bmatrix} \\ \mathbb{R}^n : \\ q_i \in \mathbb{R} \\ |q_i| \leq q_i^* \\ \text{if joint } i \text{ is revolute} \\ \text{if joint } i \text{ is prismatic} \end{array} \right\}
\]

Note that \( q_i^* \) represents the physical extension limits of prismatic joints. Uniform boundedness represented in (2)–(3) versus boundedness in (5)–(6) becomes important when establishing global stability. We will illustrate this point further with an example next. Since we are concentrating on the Hessian of potential energy in this article, we select an example control law where the uniform boundedness of the Hessian is important.

Consider the following controller PD plus simple gravity compensation originally proposed in ref. 3:

\[
\mathbf{u} = g(\mathbf{q}_d) - K_p(\mathbf{q} - \mathbf{q}_d) - K_d \dot{\mathbf{q}} \quad (7)
\]

The control objective for regulation is to achieve \( \mathbf{q} = \mathbf{q}_d \) and \( \dot{\mathbf{q}} = 0 \) at steady state. For the controller (7), this is ensured by selecting the input such that, first \( \mathbf{q} = \mathbf{q}_d \) is an equilibrium point, second the total potential energy of the robot manipulator plus the input, (1) and (7), has a global minimum at this equilibrium point, and finally the input provides sufficient damping to the system. A global minimum at the equilibrium point is ensured by making the total potential energy function globally convex. As we see shortly, to ensure the convexity of the total potential energy, a uniform bound for the Hessian is crucial.

Consider the controller (7) and the robot manipulator described by (1). The total potential energy of the robot manipulator plus input is given by

\[
V_i(\mathbf{q}) = V(\mathbf{q}) - (\mathbf{q} - \mathbf{q}_d)^T g(\mathbf{q}_d) \\
+ \frac{1}{2}(\mathbf{q} - \mathbf{q}_d)^T K_p(\mathbf{q} - \mathbf{q}_d) \quad (8)
\]

where \( V(\mathbf{q}) \) is the potential energy of the robot manipulator. By differentiating (8) with respect to \( \mathbf{q} \) we have

\[
\frac{\partial V_i(\mathbf{q})}{\partial \mathbf{q}} = g(\mathbf{q}) - g(\mathbf{q}_d) + K_p(\mathbf{q} - \mathbf{q}_d)
\]

It can be seen that \( \mathbf{q} = \mathbf{q}_d \) is a stationary point of \( V_i(\mathbf{q}) \). Differentiation with respect to \( \mathbf{q} \) once again yields

\[
\frac{\partial^2 V_i(\mathbf{q})}{\partial \mathbf{q}^2} = \frac{\partial g(\mathbf{q})}{\partial \mathbf{q}} + K_p
\]

A necessary and sufficient condition for \( \mathbf{q} = \mathbf{q}_d \) to be a local minimum is that it be a stationary point of \( V_i(\mathbf{q}) \) and that \( \frac{\partial^2 V_i(\mathbf{q})}{\partial \mathbf{q}^2} \) be positive definite at \( \mathbf{q} = \mathbf{q}_d \). To ensure a global minimum at \( \mathbf{q} = \mathbf{q}_d \), we want to select \( V_i(\mathbf{q}) \) to be strictly convex, that is, we want to select \( \frac{\partial^2 V_i(\mathbf{q})}{\partial \mathbf{q}^2} \) to be positive definite for all \( \mathbf{q} \in \mathbb{R}^n \). Therefore, we want to choose the matrix \( K_p \) such that

\[
\frac{\partial g(\mathbf{q})}{\partial \mathbf{q}} + K_p > 0 \quad \forall \mathbf{q} \in \mathbb{R}^n \quad (9)
\]
It can be easily shown that (9) is satisfied if the matrix $K_p$ is chosen to be diagonal with each entry $K_{pi}$ selected such that

$$k_{pi} > \beta$$  \hspace{1cm} (10)

where $\beta$ is a uniform bound satisfying (3). When the gain matrix $K_p$ is chosen such that (9) is satisfied, the uniqueness and global stability of the equilibrium $q = q_d$ can be achieved. However, if the Hessian is not uniformly bounded and the bound $\beta'$ of (6) was used in (10), the condition $\partial g(q)/\partial q + K_p > 0$ can only be satisfied within the robot workspace $W$. Consequently, the entire stability analysis becomes local. When a stability result is established only locally, it is necessary to know the domain of attraction. An important point to note is that even though the convexity condition is valid throughout the set $W$, the domain of attraction for the stability result could be much smaller than $W$. It is often very difficult to explicitly characterize the domain of attraction for a local stability result. Thus a global result is always more useful and the uniform boundedness is crucial in establishing global stability.

In addition to the above PD plus simple gravity compensation control law, many other control laws also require the uniform bound $\beta$ of (3). Among the control laws that use the uniform bound $\beta$ is the PI$^2$D regulator of ref. 5, the adaptive controller of ref. 6, the observer based set-point controller for robot manipulators with flexible joints proposed in ref. 7, the global regulator for flexible joint robot manipulators proposed in ref. 8, the learning gravity compensators of ref. 9, the control laws based on the energy Lyapunov function approach studied in ref. 10, and the adaptive controllers of ref. 11. In addition, the control laws of ref. 12 will also benefit from this work.

The class of robot manipulators for which the inertia matrix $D(q)$ is uniformly bounded, referred to as class $BBD$ manipulators$^b$, was characterized in ref. 13. In addition to characterizing class $BBD$, explicit expressions for the uniform bounds $\sigma_1$ and $\sigma_2$ in (2) were also derived in ref. 13. In ref. 14, we concluded that the class of robot manipulators for which a uniform bound $\gamma$ in (2) exists is class $BBF$ for which we also derived explicit expressions for $\gamma$. In previous work we also proved that for class $BB$ manipulators, a uniform bound satisfying (3) exists.$^{15,16}$ We further derived an explicit expression for the uniform bound $\beta$ in (3) in terms of constant link parameters for class $BB$ robot manipulators. Later we extended this work and fully characterized the class of robot manipulators for which the Hessian is uniformly bounded.$^{17}$ This work is an extension of the work reported in ref. 17.

The first contribution of this article is the complete characterization of the class of robot manipulators for which the Hessian is uniformly bounded. This allows one to determine by simple inspection of the joint configuration whether a uniform bound satisfying (3) exists for a given robot manipulator and thereby determine immediately whether those control laws that use the uniform bound retain globality. We refer to the class of robot manipulators with uniformly bounded Hessian as class $BBF$ robot manipulators.$^c$

Furthermore, since the gain matrix $K_p$ must be selected such that $k_{pi} > \beta$, an explicit expression for the uniform bound $\beta$ will be extremely useful in the implementation of the control law of (7). In addition to the control law of (7), the uniform bound $\beta$ of (3) is used in selecting the parameters of many other control laws also. Some of these are mentioned above. Therefore, to implement these control laws it is very useful to explicitly compute the uniform bound. The second contribution of this article is to provide, for class $BBF$ robot manipulators, an easy method to compute the explicit expression for the uniform bound $\beta$ of (3) in terms of link parameters.

Denoting the complement of class $BBF$ as class $BBF^c$, it follows that the latter consists of robot manipulators for which a uniform bound satisfying (3) does not exist. For robot manipulators of class $BBF$ even though a uniform bound satisfying (3) does not exist we can derive a bound $\beta'$ that satisfies (6). It follows that the unique equilibrium and the stability of all previously mentioned control strategies can at best be local if we use the bound $\beta'$ in place of the uniform bound $\beta$. If in the absence of a global result one chooses to apply the control laws that require the uniform bound $\beta$ of

$^b$Manipulators of this class have bounded inertia matrix $D(q)$ ($\text{Bounded D}$) from which the acronym $BBD$ is derived.

$^c$BBF is an acronym for Bounded Gravity Jacobian.
(3) to a class \(\mathcal{BFJ}\) robot manipulator, an explicit expression for the bound \(\beta'\) satisfying (6) is useful. Hence, the third contribution of this article is the derivation of an expression for the bound \(\beta'\) of (6) for class \(\mathcal{BFJ}\) robot manipulators. This bound is given in terms of the constant link parameters and upper bounds of prismatic joint variables.

The results reported in this article are valid for all serial rigid-link manipulators, and could be extended to flexible-joint and flexible-link serial manipulators. For parallel manipulators with closed kinematic chains, the equations of motion are in general defined in a compact set because of the inherent structural singularities. Consequently, for parallel manipulators, a local bound similar to \(\beta'\) in (6) could be derived (see ref. 19 for deriving a similar bound for the Rice planar parallel delta robot).

This article is organized as follows: In Section 2 we set up notation and in Section 3 we characterize class \(\mathcal{BFJ}\). We demonstrate this result with several examples. In Section 4 we derive an explicit expression for the uniform bound \(\beta\) that satisfies (3) for class \(\mathcal{BFJ}\) robot manipulators and propose an algorithm for computing \(\beta\). We illustrate the algorithm by computing \(\beta\) for the PUMA560 robot manipulator. In Section 5 we derive an explicit expression for the bound \(\beta'\) that satisfies (6) for robot manipulators of class \(\mathcal{BFJ}\) and present an example. Finally in Section 6 we draw conclusions.

### 2. KINEMATIC NOTATION

In this article we use the modified Denavit–Hartenberg (DH) convention of ref. 20 to describe the kinematics of robot manipulators (see Fig. 2). We summarize in Table I the notation used to determine the coordinate transformations.

The transformation matrix \(i^{-1}T\) from frame \((i - 1)\) to frame \(i\) is given by,

\[
i^{-1}T = \begin{bmatrix} R_{i-1}^i & d_{i-1}^i \\ 0_{1 \times 3} & 1 \end{bmatrix}\]

\[
R_{i-1}^i = R_x(\alpha_{i-1}) R_z(\theta_i)
\]

where

\[
R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix},
\]

\[
R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
d_{i-1}^i = \begin{bmatrix} a_{i-1} \\ -d_i S_{\alpha_{i-1}} \\ d_i C_{\alpha_{i-1}} \end{bmatrix}
\]

with

\[
\begin{align*}
\alpha_{i-1} &= \text{length parameter of link } i, \\
\theta_i &= \text{twist parameter of link } i, \\
d_i &= \text{offset parameter of link } (i - 1), \\
S_{\alpha_{i-1}} &= \sin(\angle_{\text{angle}}), \\
C_{\alpha_{i-1}} &= \cos(\angle_{\text{angle}}),
\end{align*}
\]

and

\[
\begin{align*}
\alpha_{i-1} &= \text{angle parameter of link } (i - 1), \\
z_i &= \text{unit vector along Z axis of frame } i \text{ in base frame }, \\
\alpha_{i-1} &= \text{unit vector along } \text{direction of the gravitational field (vertical)},
\end{align*}
\]

**Table I.** Notation for kinematics parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>Number of joints in the robot manipulator</td>
</tr>
<tr>
<td>(q_i)</td>
<td>(i)th joint variable</td>
</tr>
<tr>
<td>(q)</td>
<td>(n)-dimensional vector of joint variables</td>
</tr>
<tr>
<td>(a_i)</td>
<td>Length parameter of link (i)</td>
</tr>
<tr>
<td>(\alpha_i)</td>
<td>Twist parameter of link (i)</td>
</tr>
<tr>
<td>(d_i)</td>
<td>Offset parameter of link ((i - 1))</td>
</tr>
<tr>
<td>(\theta_i)</td>
<td>Angle parameter of link ((i - 1))</td>
</tr>
<tr>
<td>(z_i)</td>
<td>Unit vector along Z axis of frame (i) in base frame</td>
</tr>
<tr>
<td>(g)</td>
<td>Gravitational acceleration constant = 9.81 m/s(^2)</td>
</tr>
<tr>
<td>(v_i)</td>
<td>A unit vector along the direction of the gravitational field (vertical)</td>
</tr>
</tbody>
</table>
an important role. Therefore we let $c_i$ denote the position vector of the center of mass of link $i$ expressed in coordinate frame $i$ as shown in Figure 2. Since frame $i$ is attached rigidly to link $i$, $c_i$ is constant.

### 3. CHARACTERIZATION OF THE CLASS OF ROBOT MANIPULATORS WITH UNIFORMLY BOUNDED HESSIAN (CLASS $BBJJ$)

In this section we characterize class $BBJJ$ and look at several examples to illustrate this result. We begin by defining class $BBJJ$ and then prove that the necessary and sufficient condition for the Hessian of a given robot manipulator to be uniformly bounded is that it belong to class $BBJJ$.

#### 3.1. Class $BBJJ$ Robot Manipulators

**Definition 3.1.** Class $BBJJ$ consists of robot manipulators with the following joint configurations:

1. There does not exist a revolute joint $j$ and a prismatic joint $i$ such that $j < i$.
2. If there exists a revolute joint $j$ and a prismatic joint $i$ such that $j < i$, then at least one of the following conditions must be satisfied.
   a. The axis of rotation of joint $j$ and the axis of translation of joint $i$ are always parallel.
   b. The axis of rotation of joint $j$ is always parallel to $v_c$.

The following interpretation of class $BBJJ$ robot manipulators is more practical to determine whether a given robot manipulator belongs to class $BBJJ$.

**Robot Manipulators of Class $BBJJ$:** Robot manipulators of this class have any of the following joint configurations:

1. All joints are prismatic ($PR, \ldots, PR$).
2. All joints are revolute ($RR, \ldots, RR$).
3. A series of prismatic joints followed by a series of revolute joints ($PR, \ldots, PR, RR, \ldots, RR$).
4. Configurations such that for all revolute joints $j$ preceding a prismatic joint $i$, one of the following is true:
   a. The axis of rotation of joint $j$ and the axis of translation of joint $i$ are always parallel.
   b. The axis of rotation of joint $j$ is always parallel to $v_c$.

An important point to note is that the class of robot manipulators for which the inertia matrix $D(q)$ is bounded, class $BB^{13}$, is a subclass of class $BBJJ$. We now present the first major result of this article.

**Theorem 3.1:** For serial link robot manipulators described by (1), $\partial g(q)/\partial q$ is uniformly bounded if and only if the robot manipulator belongs to class $BBJJ$.

**Proof of Theorem 3.1:** See ref. 19.

#### 3.2. Examples

In this section we look at some examples to illustrate Theorem 3.1. Consider the example given in the Introduction (See Fig. 1). Since the Hessian is not uniformly bounded for this robot manipulator, according to Theorem 3.1, it should not belong to class $BBJJ$. By inspecting Definition 3.1, we see that this is indeed the case since there exists a revolute joint $j (= 1)$ and a prismatic joint $i (= 2)$ such that $j < i$ and both conditions 2a and 2b are violated. For the next example we change the relative orientations of the joints of this same robot manipulator so that it satisfies condition 2a. As we see, the Hessian becomes uniformly bounded for this case as expected from Theorem 3.1. In the final example we consider the same robot manipulator of Figure 1 and orient it such that condition 2b is satisfied. We see that this also causes the Hessian to become uniformly bounded as predicted by Theorem 3.1.

#### 3.2.1. An $PP$ Robot Manipulator with Parallel Joint Axes

For this example we consider the robot manipulator of Figure 1 and change the orientation of the axis of joint 1 with respect to the axis of joint 2 (see Fig. 3a). Note that the only revolute joint $j (= 1)$ and prismatic joint $k (= 2)$ such that $j < k$ satisfies condition 2a (since the axis of rotation of joint $j$ and the axis of translation of joint $k$ are parallel). Therefore, this robot manipulator belongs to class $BBJJ$. Next we derive an explicit expression for the Hessian for this robot manipulator. The potential energy is given by $V = m_1 gl_{c1} \cos q_1 + m_2 ga_1 \cos q_1$. Hence, the Hessian is given by,

$$
\frac{\partial g}{\partial q} = \frac{\partial^2 V}{\partial q^2} = \begin{bmatrix}
-(m_1 gl_{c1} + m_2 ga_1) \cos q_1 & 0 \\
0 & 0
\end{bmatrix}
$$
Therefore, \( \| \partial \mathbf{g} / \partial \mathbf{q} \| = |(m_1 \ell_1 + m_2 g a_1) \cos q_1| \) is uniformly bounded for this robot manipulator as predicted by Theorem 3.1.

### 3.2.2. An \( AR \) Robot Manipulator with the Revolute Joint Axis Vertical

For this example we take the same robot manipulator as in Figure 1 and orient it such that the axis of rotation of the first joint is always vertical (see Fig. 3b). Since the only revolute joint \( j \) (= 1) and prismatic joint \( k \) (= 2) such that \( j < k \) satisfies one of the required conditions, namely, the axis of rotation of joint \( j \) is always vertical, this robot manipulator also belongs to class \( BBJJ \). We consider next the Hessian for this robot manipulator. It can be seen that the center of mass locations of both links remain on the same horizontal plane as the joints are moved. Hence, for this robot manipulator, the potential energy is constant. Therefore, the Hessian becomes zero. Hence, as expected, \( \partial \mathbf{g} / \partial \mathbf{q} \) is uniformly bounded for this robot manipulator. We see from this example that in addition to the relative orientation of the joint axes, the orientation of the robot manipulator in the gravitational field also plays an important role in the uniform boundedness of the Hessian.

### 4. A UNIFORM BOUND FOR CLASS \( BBJJ \) ROBOT MANIPULATORS

In this section we present a uniform bound for the Hessian for class \( BBJJ \) robot manipulators and propose an algorithm to compute the uniform bound. This procedure is illustrated with an example.

#### 4.1. An Explicit Expression for the Uniform Bound \( \beta \)

**Theorem 4.1:** For class \( BBJJ \) robot manipulators, the uniform bound \( \beta = b \) defined as follows satisfies (3). Thus,

\[
b = \sqrt{\sum_{j=1}^{n} (\mu_j)^2 \left( \sum_{k=j+1}^{n} (\tilde{b}_k)^2 + (\tilde{b}_j)^2 \right)}
\]

(11)

Here,

\[
\mu_j = \begin{cases} 
0: & \text{if joint } j \text{ is prismatic} \\
\sin(\gamma_j): & \text{if } z_j \text{ is fixed in the base frame} \\
1: & \text{otherwise}
\end{cases}
\]

(12)
where \( \gamma_j \) is the angle between \( z_j \) and \( v_c \), and \( \bar{b}_k \) is defined as follows:

\[
\bar{b}_k = \begin{cases} 
\sum_{i=k+1}^{n} m_i \bar{d}_{k,i} + \sum_{i=k}^{n} m_i \bar{e}_{k,i} \\
0 & \text{if joint } k \text{ is revolute} \\
\text{if joint } k \text{ is prismatic}
\end{cases}
\]  

(13)

where \( m_i \) is defined as follows:

\[
m_i = \sum_{j=i}^{n} m_j
\]  

(14)

and

\[
\bar{d}_{k,i} = \begin{cases} 
\|z_k \times (R_k^{-1}d_i)\| & \text{if for all revolute joints } l \text{ such that } k < l \leq i, z_l \|z_k
\\
\|d_i\| & \text{otherwise}
\end{cases}
\]  

(15)

\[
\bar{e}_{k,i} = \begin{cases} 
\|z_k \times (R_k^{-1}c_i)\| & \text{if for all revolute joints } l \text{ such that } k < l \leq i, z_l \|z_k
\\
\|c_i\| & \text{otherwise}
\end{cases}
\]  

(16)

Proof of Theorem 4.1: See ref. 19.

4.2. Algorithm for Computing the Uniform Bound for the Hessian

We propose the following steps for computing the uniform bound \( \beta \) satisfying (3) for a robot manipulator once it is established that it is an element of class \( BBF \):

1. Assign the coordinate frames \( 0, \ldots, n \), according to the modified DH convention of ref. 20 and reviewed in Section 2.
   a. The vector \( d'_i \) is the position vector of \( O_i \) (the origin of frame \( i \)) in coordinate frame \( (i-1) \) with all prismatic joints in zero position.
   b. The vector \( c_i \) is the position vector of the center of mass of link \( i \) in coordinate frame \( i \).
2. Determine the values of \( \mu_j, j = 1, \ldots, n \), using (12).
3. Obtain the uniform bound of (11) in terms of \( \bar{b}_k, k = 1, \ldots, n \), by substituting for each \( \mu_j \).
4. Calculate the constants \( \bar{d}_{k,i}, i = k+1, \ldots, n \), and \( \bar{e}_{k,i}, i = k, \ldots, n \), corresponding to each \( \bar{b}_k \) appearing in the expression for the uniform bound using (15) and (16), respectively.
5. Calculate each \( \bar{b}_k \) and substitute in the expression for the uniform bound, Eq. (11).

4.3. Example: PUMA560

In this section we compute the uniform bound \( \beta \) of (3) for the 6 DOF PUMA560 robot manipulator (see Fig. 4). The purpose of this example is to illustrate the computational procedure proposed in the previous section and to provide a comparison of the uniform bound with the exact value of \( \| \partial g(q)/\partial q \| \) for a typical industrial robot manipulator. In the literature, there are several estimates of the dynamic parameters for the PUMA560. For this example we use the estimates given in ref. 21.

The kinematic link parameters for the PUMA560 are given in Table II which is organized as follows: Columns three to six contain the standard DH parameters while column seven contains the masses of each of the links. Finally, the last three columns contain the \( x, y, \) and \( z \) components of the position vector of the center of mass \( c_i \). The parameters \( a_i, d_i, c_{ix}, c_{iy}, \) and \( c_{iz} \) are given in meters while the parameters \( \alpha_i \) and \( \theta_i \) are given in degrees. The masses \( m_i \) are expressed in kilograms. It can be seen immediately that the PUMA belongs to class \( BBF \) since it does not contain any prismatic joints. Therefore we can use Theorem 4.1 to obtain a uniform bound for the Hessian of the PUMA560 robot manipulator. Next we will follow the procedure outlined in Section 4.2 to compute the uniform bound satisfying (3) for the PUMA560.

1. Since the axis of the first joint is always parallel to \( v_c \), \( \mu_1 = 0 \). Since none of the other joint axes are fixed in the base coordinate frame, \( \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 1 \).
2. By substituting the values above into (11), we get the following expression:

\[
b^2 = (\bar{b}_2)^2 + 2[(\bar{b}_3)^2 + (\bar{b}_4)^2 + (\bar{b}_5)^2 + (\bar{b}_6)^2]
\]

\[
+ (\bar{b}_3)^2 + 2[(\bar{b}_4)^2 + (\bar{b}_5)^2 + (\bar{b}_6)^2]
\]

\[
+ (\bar{b}_4)^2 + 2[(\bar{b}_5)^2 + (\bar{b}_6)^2] + (\bar{b}_5)^2
\]

\[
+ 2[(\bar{b}_6)^2] + (\bar{b}_6)^2
\]

\[
= (\bar{b}_2)^2 + 3(\bar{b}_3)^2 + 5(\bar{b}_4)^2 + 7(\bar{b}_5)^2 + 9(\bar{b}_6)^2
\]

(17)
3. Next we calculate the $\vec{d}_{k,i}$ corresponding to $k = 2, \ldots, 6$, from (15) as follows:

a. $\vec{d}_{2,3} = |\mathbf{z}_2 \times (R_2^3 \mathbf{d}_2^i)| = |a_2| = 0.432, \ \vec{d}_{2,4} = |\mathbf{d}_4'| = \sqrt{(a_3)^2 + (d_4')^2} = 0.433, \ \vec{d}_{2,5} = |\mathbf{d}_5'| = 0, \ \vec{d}_{2,6} = |\mathbf{d}_6'| = 0.$

b. $\vec{d}_{3,4} = |\mathbf{d}_4'| = 0.433, \ \vec{d}_{3,5} = |\mathbf{d}_5'| = 0, \ \vec{d}_{3,6} = |\mathbf{d}_6'| = 0.$

c. $\vec{d}_{4,5} = |\mathbf{d}_5'| = 0, \ \vec{d}_{4,6} = |\mathbf{d}_6'| = 0.$

d. $\vec{d}_{5,6} = |\mathbf{d}_6'| = 0.$

We next use (16) to calculate $\vec{c}_{k,i}$ for $k = 2, \ldots, 6$, as follows:

a. $\vec{c}_{2,2} = |\mathbf{z}_2 \times (R_2^3 \mathbf{c}_2)| = \sqrt{(c_{2x})^2 + (c_{2y})^2} = 0.68, \ \vec{c}_{2,3} = |\mathbf{z}_2 \times R_2^3 \mathbf{c}_3| = \sqrt{(c_{3x})^2 + (c_{3y})^2} = 0.07, \ \vec{c}_{2,4} = |\mathbf{c}_4| = 0.019, \ \vec{c}_{2,5} = |\mathbf{c}_5| = 0, \ \vec{c}_{2,6} = |\mathbf{c}_6| = 0.032.$

b. $\vec{c}_{3,3} = |\mathbf{z}_3 \times R_3^3 \mathbf{c}_3| = \sqrt{(c_{3x})^2 + (c_{3y})^2} = 0.07, \ \vec{c}_{3,4} = |\mathbf{c}_4| = 0.019, \ \vec{c}_{3,5} = |\mathbf{c}_5| = 0, \ \vec{c}_{3,6} = |\mathbf{c}_6| = 0.032.$

---

**Table II.** Kinematic link parameters of the PUMA560.

<table>
<thead>
<tr>
<th>Link $i$</th>
<th>Joint type</th>
<th>$\alpha_{i-1}$</th>
<th>$\theta_i$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$m_i$</th>
<th>$c_{i_x}$</th>
<th>$c_{i_y}$</th>
<th>$c_{i_z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Revolute</td>
<td>0</td>
<td>$q_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>Revolute</td>
<td>$-90$</td>
<td>$q_2$</td>
<td>0</td>
<td>0.244</td>
<td>17.4</td>
<td>0.068</td>
<td>0.006</td>
<td>-0.016</td>
</tr>
<tr>
<td>3</td>
<td>Revolute</td>
<td>0</td>
<td>$q_3$</td>
<td>0.432</td>
<td>-0.093</td>
<td>4.8</td>
<td>0</td>
<td>-0.070</td>
<td>0.014</td>
</tr>
<tr>
<td>4</td>
<td>Revolute</td>
<td>90</td>
<td>$q_4$</td>
<td>-0.203</td>
<td>0.433</td>
<td>0.8</td>
<td>0</td>
<td>0</td>
<td>-0.019</td>
</tr>
<tr>
<td>5</td>
<td>Revolute</td>
<td>$-90$</td>
<td>$q_5$</td>
<td>0</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>Revolute</td>
<td>90</td>
<td>$q_6$</td>
<td>0</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.032</td>
</tr>
</tbody>
</table>
To compare the uniform bound just computed with the exact value of this article, Theorem 5.1: For all robot manipulators considered in the present work, the bound expression for \(BBGG\) belonging to class 5, a bound for class 5.5. An Explicit Expression for the Bound

4. Now we substitute the above values into (13) to obtain the value of each \(\bar{b}_k\). From (14), we see that \(\bar{m}_2 = 23.4\), \(\bar{m}_3 = 6.0\), \(\bar{m}_4 = 1.2\), \(\bar{m}_5 = 0.4\), and \(\bar{m}_6 = 0.1\).

\[
\begin{align*}
\bar{b}_2 &= (\bar{m}_2 \bar{d}_{2,3} + \bar{m}_4 \bar{d}_{2,4} + \bar{m}_5 \bar{d}_{2,5} + \bar{m}_6 \bar{d}_{2,6} + 2\bar{e}_{2,2} + m_3 \bar{e}_{2,3} + m_4 \bar{e}_{2,4} + m_5 \bar{e}_{2,5} + m_6 \bar{e}_{2,6}) \bar{g} = 45.56; \\
\bar{b}_3 &= (\bar{m}_2 \bar{d}_{3,4} + \bar{m}_3 \bar{d}_{3,5} + \bar{m}_4 \bar{d}_{3,6} + m_3 \bar{e}_{3,3} + m_4 \bar{e}_{3,4} + m_5 \bar{e}_{3,5} + m_6 \bar{e}_{3,6}) \bar{g} = 8.58; \\
\bar{b}_4 &= (\bar{m}_2 \bar{d}_{4,5} + \bar{m}_4 \bar{d}_{4,6} + m_4 \bar{e}_{4,4} + m_5 \bar{e}_{4,5} + m_6 \bar{e}_{4,6}) \bar{g} = 0.03; \\
\bar{b}_5 &= (\bar{m}_6 \bar{d}_{5,6} + m_5 \bar{e}_{5,5} + m_6 \bar{e}_{5,6}) \bar{g} = 0.03; \\
\bar{b}_6 &= (m_6 \bar{e}_{6,6}) \bar{g} = 0.
\end{align*}
\]

Now by substituting in (17), we get \(\beta = b = 47.92\).

To compare the uniform bound just computed with the exact value of \(\|\partial g/\partial q\|\), we derived an exact expression for \(\partial g/\partial q\). It is independent of \(q_1\) as expected since the axis of rotation of joint 1 is always parallel to the gravitational field. To locate the point where \(\|\partial g/\partial q\|\) reaches a maximum, a complete enumeration of the robot workspace at a relatively low resolution (around 50 points for each joint resulting in a resolution of around 7\(^\circ\)) was done using Matlab. We found \(\|\partial g/\partial q\|\) to be maximum when \(q_2 = q_3 = 90^\circ\) and \(q_4 = q_5 = 180^\circ\). Figure 5 shows a plot of \(\|\partial g/\partial q\|\) and the uniform bound \(\beta\) as \(q_2\) is varied between 0 and 360° while \(q_3 = 90^\circ\) and \(q_4 = q_5 = 180^\circ\).

5. A BOUND FOR CLASS \(BBGG\) ROBOT MANIPULATORS

In this section, we present an explicit expression for the bound \(\beta^\prime\) of (6) for robot manipulators that belong to class \(BBGG\).

5.1. An Explicit Expression for the Bound \(\beta^\prime\)

Theorem 5.1: For all robot manipulators considered in this article, a bound \(\beta^\prime\) satisfying (6) is given by,

\[
\beta^\prime = b + \sum_{i=1}^{n} u_i
\]

where \(b\) is defined in (11) and

\[
u_i = \sqrt{\sum_{j=1}^{i-1} 2 \sum_{k=j+1}^{i} (\bar{\nu}_{j,k})^2 + (\bar{\nu}_{j,k})^2}:
\]

if joint \(i\) is prismatic
\(0:\) if joint \(i\) is revolute

Here,

\[
\bar{\nu}_{j,k} = \begin{cases} \bar{m}_j \sin \mu_j \mu_{j,i} & \text{if } k = i \text{ and joint } j \text{ is revolute} \\
\bar{m}_j \sin \mu_j \mu_{j,i} \bar{q}_j & \text{if both joints } j \text{ and } k \text{ are revolute} \\
0 & \text{otherwise}
\end{cases}
\]

where \(\bar{m}_i\) is defined in (14), \(\mu_j\) is defined in (12) and \(\mu_{k,i}\) is defined as follows:

\[
\mu_{k,i} = \begin{cases} |\sin(\gamma_i - \gamma_k)| & \text{if } k < l \leq i, |z_k| \quad \text{such that} \\
1 & \text{otherwise}
\end{cases}
\]

and \(\bar{q}_j\) is an upper bound for \(|q_j|\).

Proof of Theorem 5.1: See ref. 19.
5.2. Example: The Bound β' for the RP Robot Manipulator of Figure 1

In this section we choose a robot manipulator that does not belong to class \(BBGG\) to illustrate the computational procedure of the bound of Theorem 5.1. Consider the 2 DOF RP robot manipulator of Figure 1. As noted previously, this robot manipulator belongs to class \(BBGG\). The DH parameters are given in Table III which has the same format as Table II. We compute the bound \(β'\) of (18) for this robot manipulator as follows: It can be seen that to compute the bound \(β'\), we need to calculate \(b\) and \(u_i\), \(i = 1, 2\). We first calculate \(b\) following the procedure suggested in Section 4.2.

1. Since joint 2 is prismatic, \(μ_2 = 0\) and since joint 1 is revolute and \(γ_1 = 90°, \mu_1 = 1\).
2. By substituting the values above into (11), we get the following expression:
   \[ b = \sqrt{(\bar{b}_1)^2 + 2(\bar{b}_2)^2} \] (22)

3. From (15) we get \(\bar{d}_{1,2} = ||d|| = 0\). We use (16) next to obtain \(\bar{e}_{1,1} = ||z_1 \times c_1|| = ||l_{11}|| = l_{11}, \bar{e}_{1,2} = ||z_1 \times c_2|| = ||l_{21}|| = l_{21}, and \(\bar{e}_{2,2} = ||z_2 \times c_2|| = 0\).
4. Now we substitute the values above into (13) to obtain \(\bar{b}_1 = m_2 g l_{1,2} + m_1 g l_{1,1} + m_2 g l_{1,2} = m_1 g l_{c1} + m_2 g l_{c1}\) and \(\bar{b}_2 = m_2 g l_{c2} = 0\). Now by substituting in (22), we get
   \[ b = \sqrt{m_1 g l_{c1} + m_2 g l_{c2}} \]

Since joint 1 is revolute \(u_1 = 0\). We calculate \(u_2\) next using (19). From (21) we see that \(μ_{1,2} = 1\) and \(μ_{2,2} = 0\). Using (20) we obtain the following:

1. \(\bar{u}_{1,1}^2 = m_2 g μ_1 l_{c1} \bar{q}_2 = m_2 g \bar{q}_2\).
2. \(\bar{u}_{1,2}^2 = m_2 g μ_1 μ_{2,2} = 0\).
3. \(\bar{u}_{2,2}^2 = m_2 g μ_2 μ_{2,2} = 0\).

Now by substituting in (19), we get
\[ u_2 = \sqrt{(\bar{u}_{1,1})^2 + 2(\bar{u}_{1,2})^2 + (\bar{u}_{2,2})^2} = \sqrt{(m_2 g \bar{q}_2)^2} = m_2 g \bar{q}_2 \]

Finally by substituting in (18), we obtain
\[ β' = m_1 g l_{c1} + m_2 g l_{c2} + m_2 g \bar{q}_2 \] (23)

As in the previous example, to compare the bound with the exact value of \(\|\partial g(q) / \partial q\|\), we use the expression for \(\partial g(q) / \partial q\) in (4) to generate a plot of \(\|\partial g(q) / \partial q\|\) and the bound \(β'\) given in (23) as \(q_i\) is varied from \(-180\) to \(180°\) and \(q_2\) is varied between 0 and 1.5 m (see Fig. 6). We chose \(m_1 = 5, m_2 = 4, l_{11} = 0.25, l_{22} = 0.2, and \bar{q}_2 = 1.5\). Note that this bound is only valid locally in the region satisfying \(|q_2| ≤ \bar{q}_2\).

6. CONCLUSIONS

Many theoretical results on motion regulation and tracking require a uniform bound for the Hessian. However, not all joint configurations of robot manipulators insure the existence of such a uniform bound. Therefore, for the implementation of these control laws it is important to characterize the class of robot manipulators for which the Hessian is uniformly bounded (class \(BBGG\)). The first contribution of this article was the full characterization of class \(BBGG\). The requirement for a robot manipulator to belong to class \(BBGG\) depended on both the relative orientations of the joints as well as the orientation of the robot manipulator in the gravity field. Three examples were presented to illustrate this result.

The selection of parameters of those control laws that require a uniform bound for the Hessian, is based on the value of the uniform bound. Therefore, an explicit expression for the uniform bound will be useful in the implementation of these control laws. The second contribution of this article was the derivation of an explicit expression for the uniform bound for class \(BBGG\) robot manipulators. This bound is given in terms of constant kinematic and inertial link parameters of the robot manipulator. We presented a simple algorithm for computing this bound and illustrated it by computing the uniform bound for the PUMA560 robot manipulator.

The third contribution of this article was an explicit expression of a bound for the Hessian for

<table>
<thead>
<tr>
<th>Link i</th>
<th>Joint type</th>
<th>(α_{i-1})</th>
<th>(θ_i)</th>
<th>(a_{i-1})</th>
<th>(d_i)</th>
<th>(m_i)</th>
<th>(c_{ix})</th>
<th>(c_{iy})</th>
<th>(c_{iz})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Revolute</td>
<td>0</td>
<td>(q_1)</td>
<td>0</td>
<td>0</td>
<td>(m_1)</td>
<td>0</td>
<td>(l_{c1})</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Prismatic</td>
<td>90</td>
<td>0</td>
<td>(q_2)</td>
<td>(m_2)</td>
<td>0</td>
<td>(l_{c2})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
class $BBGJ$ robot manipulators that is valid within the physical workspace of the manipulator. This bound is given in terms of constant link parameters and upper bounds for prismatic joint variables, and it is useful in the design and gain synthesis of local control laws. This result was also illustrated with an example.

In conclusion, this article clarifies important issues related to the application and implementation of several control laws that rely on the boundedness of the Hessian of potential energy of robot manipulators. The characterization of class $BBGJ$ is very important as it enlarges the very restricted class of robot manipulators typically assumed in the literature in the design of many control laws.

The easily computable explicit uniform bounds $\beta$ in (3) and $\beta'$ in (6) proposed in this article, as well as our previous work on explicit expressions for $\sigma_1$ and $\sigma_2$ in (2) (see ref. 13) and for $\gamma$ in (2) (see ref. 14), will be extremely useful in the synthesis of controller gains of several existing results in the literature.

REFERENCES


