In the control of robot manipulators, it is customary to assume that the eigenvalues of the inertia matrix are uniformly bounded from below and above. However, in this article it is shown that not all manipulators possess this property. The class of serial robot manipulators with bounded inertia matrix, referred to as class BB DD manipulators, is completely characterized and it is shown that it includes manipulators with nontrivial joint configurations. For manipulators of this class, easily computable uniform bounds for the minimum and maximum eigenvalues of the inertia matrix are provided. © 1998 John Wiley & Sons, Inc.
1. INTRODUCTION

The standard model for the dynamics of an $n$-link rigid robot manipulator is given by

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$  \hspace{1cm} (1)

where the $n$-vectors $q$, $\dot{q}$, and $\ddot{q}$ represent the link angles, velocities and accelerations, respectively, $D(q)$ is the $n \times n$ inertia matrix, $C(q, \dot{q})\dot{q}$ represents the Coriolis and centrifugal terms, $g(q)$ represents the gravitational terms, and $u$ is the input vector.

Although the equations given in (1) are complex nonlinear equations, a remarkable body of results is now available for the trajectory control of such systems. The success of these results stem from a well understanding of the properties and structure of the kinematics and dynamics models of robot manipulators. An interesting study of the properties of the terms in (1) and their functional dependence can be found in refs. 2 and 3. Our focus in this article is on the inertia matrix $D(q)$ which has two important properties that are exploited in a fundamental way in the design and analysis of control laws for (1), namely, its positive definiteness\textsuperscript{2,4} and its boundedness.

Positive definiteness and uniform boundedness relate to the minimum and maximum singular values of the inertia matrix. In much of the existing robot control literature, it is typically assumed that there exist positive constants $\sigma_1$ and $\sigma_2$ such that

$$0 < \sigma_1 \leq \|D(q)\| \leq \sigma_2 < \infty.$$  \hspace{1cm} (2)

The existence of the above positive constants, namely, $\sigma_1$ and $\sigma_2$, is the basis of gain controller design and global Lyapunov stability development of many of the existing popular manipulator control laws (see, for example, refs. 1, 5-11). One typical example that illustrates the use of the bounds in (2) is in the stability analysis and design of robust controllers in which it is assumed, among other conditions, that\textsuperscript{1,9}

$$\|D(q)^{-1} \dot{D}(q) - I\| \leq \nu < 1$$

for some constant $\nu$ and for all $q$.

Here, $\dot{D}$ is the nominal or computed version of $D$, and $I$ is the identity matrix. It turns out\textsuperscript{1} that the above mentioned assumption is automatically satisfied when $\nu$ is chosen using (2) as follows:

$$\nu = \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1}.$$  \hspace{1cm} (2a)

The constant coefficient $\nu$ is used, among other bounds, in both the stability analysis as well as in computing the robust controller gains (see, for example, refs. 1 and 10).

Since the existence of the uniform bounds $\sigma_1$ and $\sigma_2$ in (2) is essential to many existing control laws, the class of manipulators considered in the literature is normally restricted to those which have only revolute joints ($\mathop{\mathcal{R}}\mathop{\mathcal{R}} \cdots \mathop{\mathcal{R}}$)\textsuperscript{b} for which (2) is clearly satisfied. Since the restriction is basically made for convenience, it stands to reason to question the assumption made almost universally about restricting analysis to this class. There is a need, therefore, to examine the question whether robot manipulators with mixed (revolute and prismatic) joint configurations could also satisfy (2). Furthermore, it is very important to devise general and

\textsuperscript{a} The following standard notation and terminology is used: $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^n$ denotes the usual $n$-dimensional vector space over $\mathbb{R}$ endowed with the Euclidean norm $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ matrices with real elements. For $M \in \mathbb{R}^{n \times m}$, $\|M\|$ is the induced 2-matrix norm of $M$ corresponding to the Euclidean vector norm on $\mathbb{R}^n$.

\textsuperscript{b} The case in which all joints are prismatic ($\mathop{\mathcal{P}}\mathop{\mathcal{P}} \cdots \mathop{\mathcal{P}}$) is trivial because the corresponding inertia matrix $D$ is constant.
easy methods to explicitly compute the uniform bounds $\sigma_1$ and $\sigma_2$ in (2).

Clearly, a manipulator with a PR joint configuration ensures (2) as the inertia matrix is independent of the first (prismatic in this case) joint variable. Hence, the class of manipulators satisfying (2) is larger than that assumed in the literature. On the other hand, not all joint configurations give rise to (2). Consider for example a two DOF (degrees of freedom) manipulator with one revolute joint followed by a prismatic joint as shown in Figure 1. The corresponding inertia matrix is

$$D(q_2) = \begin{bmatrix} m_2 q_2^2 + I_1 + I_2 & 0 \\ 0 & m_2 \end{bmatrix}$$

where $m_2$, $I_1$, and $I_2$ are the mass of link 2, inertia of link 1, and inertia of link 2, respectively. Note that $d_{12} = (m_2 q_2^2 + I_1 + I_2) \to \infty$ as $q_2 \to \infty$. Hence, there is no constant $\sigma_2$ such that (2) is satisfied for all possible values of $q_2$.

In this article, we completely characterize the class of robot manipulators for which each of the elements of the inertia matrix is bounded. For this class, referred to as class BB manipulators, we prove the existence of the uniform bounds $\sigma_1$ and $\sigma_2$ in (2). Furthermore, we derive easily computable explicit expressions for $\sigma_1$ and $\sigma_2$ in terms of kinematic and inertial link parameters.

This article is organized as follows. In section 2, we briefly review the forward kinematics, the velocity Jacobian, and the expression of inertia matrix. Section 3 deals with the structural properties of the inertia matrix which are fundamental in the development of the rest of the article. In section 4, we completely characterize class BB manipulators which consists of manipulators for which all the elements of the inertia matrix are bounded. In section 5, we propose explicit expressions for the uniform bounds of the inertia matrix of class BB manipulators. Section 6 presents an example to illustrate the computation of the uniform bounds for a manipulator of class BB, and finally in section 7 conclusions are drawn.

2. FORWARD KINEMATICS, VELOCITY JACOBIAN, AND INERTIA MATRIX

A detailed treatment of kinematics and dynamics of serial link robot manipulators can be found in the text. In this section, we briefly review the forward kinematics of robot manipulators, the structures of the velocity Jacobian matrix and the inertia matrix using the notation used in the above reference. Our purpose with such a review is to set up notation for a further discussion about the boundedness of the inertia matrix in the rest of the paper.

Serial link robot manipulators considered in this paper consist of a set of rigid links connected together by revolute and prismatic joints. A revolute joint permits relative rotation of two links about the axis of the joint, whereas a prismatic joint allows for the linear motion of a link along an axis and the resulting motion is either an extension or retraction. By convention, an $n$ degree-of-freedom ($n$ DOF) robot has $n+1$ links numbered from 0 to $n$ with the base taken as link 0, and $n$ joints where the $i$th-joint connects link $i-1$ and link $i$. We denote by $q_i$ the $i$th-joint variable, $q_i \in \mathbb{R}$, where $q_i$ is the angle of rotation if the joint is revolute, and $q_i$ is the joint displacement if the joint is prismatic. We also denote by $\mathbf{q}$ the vector of joint variables $q_i$, $i = 1,2,\ldots,n$.

Using the Denavit–Hartenberg frame assignment, each link $i$ is characterized by four kinematics parameters: the length $a_i$, the twist $\alpha_i$, the offset $d_i$, and the angle $\theta_i$. Among the four parameters, only the angle $\theta_i$ is variable if the joint is revolute, and

\[\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \text{and} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \text{for} \quad i = 1,2,\ldots,n.\]
only \( d_i \) is variable if the joint is prismatic. Given these link parameters, we solve the so called forward kinematics problem which consists of determining the position and orientation of the end effector in base frame as a function of joint positions. For an \( n \) DOF robot manipulator, the transformation matrix \( T_0^j \in \mathbb{R}^{4\times 4}, 1 \leq j \leq n, \) defined by

\[
T_0^j = \prod_{i=1}^j A_i
\]

(3)

gives the position and orientation of the tip of link \( j \) expressed in base coordinates. The matrix \( A_i, i \in \{1,2,\ldots, n\} \) is the homogeneous transformation given by

\[
A_i = \begin{bmatrix}
R_i^{i-1} & d_i^{i-1} \\
0_{1\times 3} & 1
\end{bmatrix}
\]

(4)

\[
R_i^{i-1} = 
\begin{bmatrix}
C_{\theta_i} & -S_{\theta_i}C_{a_i} & S_{\theta_i}S_{a_i} \\
S_{\theta_i} & C_{\theta_i}C_{a_i} & -C_{\theta_i}S_{a_i} \\
0 & S_{a_i} & C_{a_i}
\end{bmatrix}
\]

(5)

\[
d_i^{i-1} = 
\begin{bmatrix}
a_iC_{\theta_i} \\
a_iS_{\theta_i} \\
d_i
\end{bmatrix}
\]

(6)

where \( C_{\text{angle}} = \cos(\text{angle}) \), and similarly \( S_{\text{angle}} = \sin(\text{angle}) \). In light of (4), the transformation matrix \( T_0^j \) given in (3) takes the following form:

\[
T_0^j = \left[ \prod_{i=1}^j R_i^{i-1} \right] _{0\times 3} + \sum_{i=1}^{j-1} \left[ \prod_{k=1}^i R_k^{k-1} \right] d_i^{i+1}
\]

(7)

We define the following quantities for future use in the paper:

\[
R_j = \prod_{i=1}^j R_i^{i-1}
\]

(8)

\[
o_j = d_0 + \sum_{i=1}^{j-1} R_i d_i^{i+1}
\]

(9)

\[
x_0 = [1\ 0\ 0]^T
\]

(10)

\[
z_0 = [0\ 0\ 1]^T
\]

(11)

\[
z_j = R_j z_0
\]

(12)

\[
x_j = R_j x_0
\]

(13)

The velocity relationships between joint velocities \( \dot{q} \) and linear and angular velocities \( (v \in \mathbb{R}^3 \) and \( w \in \mathbb{R}^3 ) \) of the end effector for an \( n \) DOF robot manipulator is given by the Jacobian \( J(q) \in \mathbb{R}^{6\times n} \) as follows for \( i = 1,2,\ldots, n \)

\[
\begin{bmatrix} v \\ w \end{bmatrix} = J(q) \dot{q}, \quad J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}, \quad J_v, J_w \in \mathbb{R}^{3\times n}
\]

(14)

The columns of the Jacobian are computed as follows:

\[
J_v = \begin{cases}
[z_{i-1} \times (o_i - o_{i-1})] & \text{if joint } i \\
[z_{i-1}] & \text{if joint } i \\
0 & \text{is prismatic}
\end{cases}
\]

(15)

The Jacobian equation (14) is also useful to compute the Jacobian with respect to the center of mass of link \( k \), denoted by \( J_{c_k} \in \mathbb{R}^{6\times n} \), as follows:

\[
J_{c_k} = 
\begin{bmatrix}
J_{v_{c_k}} \\
J_{w_{c_k}}
\end{bmatrix} = 
\begin{bmatrix}
J_{v_{c_k,1}} & J_{v_{c_k,2}} & \cdots & J_{v_{c_k,k}} & 0 & \cdots & 0 \\
J_{w_{c_k,1}} & J_{w_{c_k,2}} & \cdots & J_{w_{c_k,k}} & 0 & \cdots & 0
\end{bmatrix}
\]

(16)

where \( i = 1,2,\ldots, k \)

\[
J_{v_{c_k,i}} = \begin{cases}
[z_{i-1} \times (o_{c_k} - o_{i-1})] & \text{if joint } i \\
[z_{i-1}] & \text{is revolute} \\
0 & \text{is prismatic}
\end{cases}
\]

(17)

In general, the vector \( o_{c_k} \) must be computed as it is not given directly by the \( T \) matrices. For simplicity, and without loss of generality, we make the following assumption:

**Assumption 2.1:** We assume that for a revolute joint \( j \), the center of mass of link \( j \), is located along the same axis as that of the length parameter \( a_j \). It follows that the vector \( o_j \) is obtained from the vector \( o_i \) by substituting \( a_j \) by \( l_i \), where the latter is the distance from the origin of the frame of joint \( j \) to the center of mass of link \( j \). For a prismatic joint \( j \), we also assume that the center of mass of link \( j \) is located along the same axis as that of the variable \( d_j \). Hence, the vector \( o_{c_j} \) is obtained from the
vector \( \mathbf{o}_{ij} \) by substituting the variable \( d_j \) by a quantity proportional to it, say \( \tilde{d}_j \). We therefore write

\[
\mathbf{o}_{ij} = \mathbf{d}_0^1 + \sum_{i=1}^{j-2} \mathbf{R}_i \mathbf{d}_{i+1} + \mathbf{R}_{j-1} \tilde{d}_{j-1} \tag{18}
\]

where

\[
\tilde{d}_{j-1} = \begin{cases} 
\begin{bmatrix} \mathbf{l}_j \mathbf{c}_{\theta_j} \\
\mathbf{l}_j \mathbf{s}_{\theta_j} \\
\mathbf{d}_j \\
\mathbf{a}_j \mathbf{c}_{\theta_j} \\
\mathbf{a}_j \mathbf{s}_{\theta_j} \\
\mathbf{d}_{ij} 
\end{bmatrix} & \text{if joint } j \text{ is revolute} \\
\end{cases}
\]

Note 2.1: For manipulators that do not satisfy Assumption 2.1, the vector \( \mathbf{o}_{ij} \) will have an expression similar to that in (18) with an additional constant term. The latter term will only influence the magnitude of the uniform bounds \( \sigma_1 \) and \( \sigma_2 \) and will not alter the analysis and results that follow.

A detailed analysis in ref. 1 shows that for an \( n \) DOF robot manipulator, the \( n \times n \) inertia matrix is given by

\[
\mathbf{D}(\mathbf{q}) = \sum_{i=1}^{n} \left[ m_i \mathbf{J}_{\mathbf{v}_i}^T \mathbf{J}_{\mathbf{v}_i} + \mathbf{J}_{\mathbf{w}_i}^T \mathbf{R}_i \mathbf{J}_{\mathbf{w}_i} \right] (\mathbf{q}) \tag{20}
\]

where \( m_i \) is the mass of link \( i \), and \( \mathbf{J}_i \in \mathbb{R}^{3 \times 3} \) is the positive definite inertia matrix of link \( i \) evaluated with respect to a coordinate frame parallel to the frame at joint \( i \) but whose origin is at the center of mass of link \( i \). It is given by

\[
\mathbf{J}_i = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\
I_{xy} & I_{yy} & I_{yz} \\
I_{xz} & I_{yz} & I_{zz} 
\end{bmatrix} \tag{21}
\]

We close this review section by highlighting the following properties and relationships which will be exploited in the remainder of the article:

\[
x_j \times z_j \text{ are unit vectors } \Rightarrow ||x_j \times z_j|| = ||x_j \times z_j|| = 1 \tag{22}
\]

\[
\mathbf{R}_j \text{ is orthogonal } \Rightarrow \mathbf{R}_j^{-1} = \mathbf{R}_j^T 
\]

and \( \mathbf{R}_j \) has unity eigenvalues \( \tag{23} \)

\[
\mathbf{o}_j - \mathbf{o}_{j-1} = \mathbf{R}_{j-1} \mathbf{d}_{j-1}, \quad \text{and } \quad \mathbf{o}_{ij} - \mathbf{o}_{ij-1} = \mathbf{R}_{j-1} \mathbf{d}_{j-1} \tag{24}
\]

\[
\mathbf{d}_{j-1} = d_j \mathbf{z}_0 + a_j \mathbf{R}_{j-1}^T \mathbf{x}_0 \tag{25}
\]

3. STRUCTURE OF THE INERTIA MATRIX

To prepare the background for further analysis, we highlight some useful properties of the inertia matrix first. We have the following fact:

**Fact 3.1:** The inertia matrix \( \mathbf{D}(\mathbf{q}) \) given by (20) can be written as follows:

\[
\mathbf{D} = \sum_{k=1}^{n} \mathbf{D}(k) \tag{26}
\]

with the following properties

- **(P1):** \( \mathbf{D}(k) \) is an \( n \times n \) symmetric positive semi-definite (PSD) matrix, \( k = 1, 2, \ldots, n \).
- **(P2):** The elements \( d_{ij}(k) \) of the matrix \( \mathbf{D}(k) \) satisfy \( d_{ij}(k) = 0 \) for \( i > j \) or \( j > i \), \( k = 1, 2, \ldots, n \), and,
- **(P3):** \( d_{ik}(k) > 0 \). In fact,

\[
d_{ik}(k) = \begin{cases} 
m_k & \text{if joint } k \text{ is prismatic} \\
\frac{m_k I_{xx}^2 + I_{yy}^2 + I_{zz}^2 + 2 I_{yz} I_{zy}^2}{a_k^2} & \text{if joint } k \text{ is revolute.} 
\end{cases} \tag{27}
\]

**Remark 3.1:** The physical interpretation of the element \( d_{ik}(k) \) is as follows. For a prismatic joint \( k \) it is the mass \( m_k \), and for a revolute joint, it is the inertia of link \( k \) about its axis of rotation.

**Proof of Fact 3.1:** From equation (20), we have

\[
\mathbf{D}(k) = m_k \mathbf{J}_{\mathbf{v}_k}^T \mathbf{J}_{\mathbf{v}_k} + \mathbf{J}_{\mathbf{w}_k}^T \mathbf{R}_k \mathbf{J}_{\mathbf{w}_k} \tag{28}
\]

It is clear from expression (28) that \( \mathbf{D}(k) \) is symmetric. Since it is the sum of two PSD terms, it follows that \( \mathbf{D}(k) \) is itself PSD. Hence property **(P1)** is veri-
Using (12) and (23), we write

$$d_{kk}(k) = m_k J^T_{r_{v_{ij}}} + J^T_{w_{v_{ij}}} R_k I_k R_k^T J_{v_{ij}}$$

(29)

If joint $k$ is prismatic, then $J^T_{r_{v_{ij}}} = z_{k-1}$ and $J^T_{w_{v_{ij}}} = 0$. Since in addition $z$ is a unit vector, it follows that

$$d_{kk}(k) = m_k (z_{k-1} \times (o_{c_i} - o_{k-1}))$$

(30)

If joint $k$ is revolute, then $J^T_{r_{v_{ij}}} = z_{k-1} \times (o_{c_i} - o_{k-1})$ and $J^T_{w_{v_{ij}}} = z_{k-1}$. Consequently,

$$d_{kk}(k) = m_k \left[ z_{k-1} \times (o_{c_i} - o_{k-1}) \right]^T$$

$$\cdot \left[ z_{k-1} \times (o_{c_i} - o_{k-1}) \right] + z_{k-1}^T R_k I_k R_k^T z_{k-1}$$

(31)

Using (12) and (23), we write

$$z_{k-1}^T R_k I_k R_k^T z_{k-1}$$

$$= z_0^T R_k^T \left( R_k I_k R_k^T \right) R_k^T z_0$$

$$= z_0^T R_k^T R_{k-1} R_{k-1} I_k R_k^T R_{k-1} R_k^T z_0$$

$$= z_0^T R_k^T I_k R_k^T z_0$$

$$= I_{yy} S_{a_i}^2 + I_{zz} C_{a_i}^2 + 2 I_{yz} S_{a_i} C_{a_i}$$

(32)

We now use (12), (22), and (24) and write

$$\left\| z_{k-1} \times (o_{c_i} - o_{k-1}) \right\|^2$$

$$= \left\| z_{k-1} \times R_{k-1} d_{kk} z_{k-1} \right\|^2$$

$$= \left\| z_{k-1} \times R_{k-1} (d_{kk} z_{k-1} + l_{c_i} R_{k-1} x_0) \right\|^2$$

$$= \left\| d_{kk} z_{k-1} \times z_{k-1} + l_{c_i} z_{k-1} \times x_0 \right\|^2$$

$$= I_{c_i}$$

(33)

In the above equation, we used the fact that in the Denavit–Hartenberg representation $\| z_{k-1} \times x_0 \| = 1$ [see (22)]. Using (32) and (33), (31) becomes

$$d_{kk}(k) = m_k I_{c_i}^2 + I_{yy} S_{a_i}^2 + I_{zz} C_{a_i}^2 + 2 I_{yz} S_{a_i} C_{a_i}$$

(34)

With equations (30) and (34), P3 is now verified. 

4. CLASS $\mathcal{B}$: THE CLASS OF ROBOT MANIPULATORS WITH BOUNDED INERTIA MATRIX

In this section, we discuss the class of robot manipulators for which the elements of the inertia matrix are uniformly bounded. We first present the following definitions.

**Definition 4.1:** The element $m_{ij}(q)$ of the matrix $M(q)$ is bounded if there exists a constant $c < \infty$ such that $|m_{ij}(q)| \leq c \forall q \in \mathbb{R}^n$.

**Definition 4.2:** A matrix $M(q)$ is bounded if all its elements are bounded $\forall q \in \mathbb{R}^n$. The matrix $M(q)$ is unbounded if at least one of its elements is not bounded.

**Definition 4.3:** A manipulator is of class $\mathcal{B}$ if its inertia matrix is bounded.

We have the following result.

**Theorem 4.1:** The inertia matrix $D(q)$ is unbounded if and only if in the joint configuration of the robot manipulator there exist a revolute joint $k$ and a prismatic joint $j$, $k < j$, such that

$$z_{k-1} \times z_{j-1} \neq 0$$

(35)

**Proof of Theorem 4.1:** Recall that the inertia matrix is given by

$$D(q) = \sum_{k=1}^{n} D(k)$$

(36)

$$D(k) = m_k J^T_{r_{v_{ij}}} + J^T_{w_{v_{ij}}} R_k I_k R_k^T J_{w_{v_{ij}}}$$

(37)

We first note that the elements of the vector $q$ consist of either link angles $\theta$ or link offsets $d$. The variable $\theta$ appears only in trigonometric form, that is, only $\cos(\theta)$ and $\sin(\theta)$ appear in $D(q)$. Since the cosine and sine functions are bounded functions, the variable $\theta$ cannot be the source of unboundedness of the inertia matrix. On the other hand, polynomials in the variable $d$ appear in the expression of the inertia matrix, and hence it is the only possible way to make the inertia matrix unbounded for unbounded $d$. By further examining the expression of the inertia matrix above, we note that the quantities that depend on $q$ are $R_k$, $J_{v_{ij}}$, and $J_{w_{ij}}$. The elements of $R_k$ are always bounded since they consist of trigonometric functions of the angle $\theta$. The
columns of $J_{x_j}$ [see (17)] are either zero vectors or the $z$ vectors. In both cases, the elements of $J_{x_j}$ are bounded since the $z$ vectors (unit vectors) consist of trigonometric functions of the angle $\theta$ and they never include the variable $d$. Consequently, the only source of the variable $d$ in the inertia matrix originates from the matrices $J_{x_k}$ and more specifically from the $o$ vectors. If $J_{x_j}$ is unbounded for unbounded $d_j$, then $\partial J_{x_j}/\partial d_j \neq 0$. With this background, we now prove the theorem.

**"if" Part:** If there exist a revolute joint $k$ and a prismatic joint $j$, $k < j$, such that $z_{k-1} \times z_{j-1} \neq 0$, then the inertia matrix is unbounded.

Proof of "if" Part: If there exist a revolute joint $k$ and a prismatic joint $j$, $k < j$, such that $z_{k-1} \times z_{j-1} \neq 0$, then we claim that at least one of the elements of the inertia matrix would be unbounded and the proof of the "if" part would be completed. To prove the claim, we write

$$J^k_{v_j} = z_{k-1} \times (o_{c_j} - o_{k-1})$$

$$= z_{k-1} \times (o_{c_j} - o_{j-1}) + z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$= z_{k-1} \times R_{j-1} d_j + z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$= z_{k-1} \times R_{j-1} (d_j z_0 + a_j R_{j-1} x_0)$$

$$+ z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$= d_j z_k + a_j z_{k-1} \times x_j$$

$$+ z_{k-1} \times (o_{j-1} - o_{k-1})$$

(38)

Note that $\partial J^k_{v_j}/\partial d_j = z_{k-1} \times z_{j-1}$. Since by assumption $z_{k-1} \times z_{j-1} \neq 0$ and from Assumption 2.1, $d_j$ is proportional to $d_j$, it follows that $J^k_{v_j}$ is unbounded.

**"only if" Part:** If the inertia matrix is unbounded then there exist a revolute joint $k$ and a prismatic joint $j$, $k < j$, such that $z_{k-1} \times z_{j-1} \neq 0$.

Proof of "only if" Part: If an element of the inertia matrix is unbounded, then there exists at least one vector $J^k_{v_j}, k < l$, which is unbounded. Joint $k$ cannot be prismatic as if it were prismatic, then $J^k_{v_j} = z_{k-1}$ would be bounded. Hence, joint $k$ has to be revolute.

Now suppose that joint $l$ is also revolute, then for $J^k_{v_j}$ to be unbounded, there must exist a prismatic joint $j,k < l$ such that the unboundedness of $J^k_{v_j}$ is due to the presence of $q_j = d_j$. Hence, we write

$$J^k_{v_j} = z_{k-1} \times (o_{c_j} - o_{k-1})$$

$$= z_{k-1} \times (o_{c_j} - o_{j-1}) + z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$+ z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$= z_{k-1} \times (o_{c_j} - o_{j-1}) + z_{k-1} \times R_{j-1} d_1 z_0 + a_j R_{j-1} x_0)$$

$$+ z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$= z_{k-1} \times (o_{c_j} - o_{j-1}) + z_{k-1} \times (o_{j-1} - o_{k-1})$$

$$= z_{k-1} \times (o_{c_j} - o_{j-1}) + d_j z_{k-1} \times z_{j-1} + a_j z_{k-1} x_j$$

$$+ z_{k-1} \times (o_{j-1} - o_{k-1})$$

(39)

Note that $\partial J^k_{v_j}/\partial d_j = z_{k-1} \times z_{j-1}$. Since $J^k_{v_j}$ is unbounded due to the presence of $d_j$, i.e., $\partial J^{k}_{v_j}/\partial d_j \neq 0$, it follows that $z_{k-1} \times z_{j-1} \neq 0$.

If $j = l$, i.e., joint $l$ is prismatic, then $J^{k}_{v_j}$ satisfies (38). Since $\partial J^{k}_{v_j}/\partial d_j = \partial d_{c_j}/\partial d_j (z_{k-1} \times z_{j-1})$ and $d_{c_j}$ is proportional to $d_j$, the fact that $J^{k}_{v_j}$ is unbounded due to the presence of $d_j$ implies that $z_{k-1} \times z_{j-1} \neq 0$.

Hence we have proved that there must exist a revolute joint $k$ and a prismatic joint $j,k < j$, such that $z_{k-1} \times z_{j-1} \neq 0$ which completes the “only if” part of the proof and hence the whole proof.

**Corollary 4.1:** If a joint configuration $R$ results in a bounded inertia matrix, then starting from the base, the joint configuration $R^n R^m$ also results in a bounded inertia matrix, where $R^n$ and $R^m$ denote a series of $n$-prismatic and $m$-revolute joints, respectively.

Our result in this section is summarized as follows:

**4.1. Result:** Joint Configurations for Bounded Inertia Matrix

Robot manipulators of class $\mathcal{BD}$ have one of the following joint configurations:

1. All joints are prismatic ($R = \text{null configuration}$, $m = 0$).
2. All joints are revolute ($R = \text{null configuration}$, $n = 0$).
3. A series of prismatic joints followed by a series of revolute joints (0 = null configuration).

4. Configurations where the axis of translation of each prismatic joint $j$ is parallel to all preceding revolute joints $k$, that is, $z_{k-1} \times z_{j-1} = 0$.

5. **UNIFORM BOUNDEDNESS OF THE INERTIA MATRIX**

In this section, we give explicit uniform bounds for the inertia matrix of robot manipulators with bounded inertia matrix. We first present some definitions and background material.

### 5.1. Background

**Definition 5.1:** The max-bound of a real PSD matrix $M$ is a mapping $\|M\|_{\max} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^+$ defined by $\|M\|_{\max} = \max_{x \in \mathbb{R}^n} x^T M x$.

**Definition 5.2:** The min-bound of a real PSD matrix $M$ is a mapping $\|M\|_{\min} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^+$ defined by $\|M\|_{\min} = \min_{x \in \mathbb{R}^n} x^T M x$.

**Remark 5.1:** The max-bound defined above is the induced norm defined for matrices. From the eigenvalue viewpoint, the max-bound is the maximum eigenvalue and the min-bound is the minimum eigenvalue.

**Definition 5.3:** A matrix $M(q)$ is said to be uniformly bounded from above if $\exists$ a constant $c < \infty$ such that $\|M(q)\|_{\max} \leq c \forall q \in \mathbb{R}^n$.

**Definition 5.4:** A matrix $M(q)$ is said to be uniformly bounded from below if $\exists$ a constant $c > 0$ such that $\|M(q)\|_{\min} \geq c \forall q \in \mathbb{R}^n$.

**Proposition 5.1:** If $\sum_{i=1}^n x_i = T, \ x_i \geq 0, \ \forall i$, then $\prod_{i=1}^n x_i \leq T^n/n^n$.

**Proof of Proposition 5.1:** Consider maximizing $\prod_{i=1}^n x_i$ with constraint $\sum_{i=1}^n x_i = T$. Using Lagrange multiplier optimization method, we maximize the function $J = \prod_{i=1}^n x_i + \mu(\sum_{i=1}^n x_i - T)$. At the maximum, $\partial J/\partial x_j = \prod_{i=1,j}^n x_i + \mu = 0, \ \forall j$. It follows that $x_1 = x_2 = \cdots = x_n = T/n$ and the maximum value is $T^n/n^n$.

We now present this useful lemma:

**Lemma 5.1:** The min-bound and max-bound of a real $PD$ $n \times n$ matrix $M (n > 1)$ with determinant $\Delta$ and trace $\mathcal{T}$ obey the inequalities

$$\|M\|_{\min} > \frac{(n-1)^{\frac{n-1}{2}} \Delta}{\mathcal{T}^{\frac{n-1}{2}}}$$

$$\|M\|_{\max} < \mathcal{T} - \frac{(n-1)^{\frac{n-1}{2}} \Delta}{\mathcal{T}^{\frac{n-1}{2}}}$$

**Proof of Lemma 5.1:** Consider the eigenvalues of the PD matrix $M$ ordered as follows: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Hence $\|M\|_{\min} = \lambda_1, \ |M|_{\max} = \lambda_n, \ \Delta = \prod_{i=1}^n \lambda_i, \ \mathcal{T} = \sum_{i=1}^n \lambda_i$. It follows that $\Delta/\lambda_i \leq \mathcal{T}/(n-1)^{\frac{n-1}{2}}$. Since $\lambda_1 > 0$, $(\mathcal{T}/\lambda_i) > (n-1)^{\frac{n-1}{2}}$. Hence, $\Delta/\lambda_1 < \mathcal{T}/(n-1)^{\frac{n-1}{2}}$ and consequently $\|M\|_{\min} = \lambda_1 > (n-1)^{\frac{n-1}{2}} \Delta/\mathcal{T}^{\frac{n-1}{2}}$. We now write $\mathcal{T}/\lambda_n = \sum_{i=1}^{n-1} \lambda_i \geq (n-1)\lambda_1 > (n-1)(n-1)^{\frac{n-1}{2}} \Delta/\mathcal{T}^{\frac{n-1}{2}}$. Hence, $\|M\|_{\max} = \lambda_n < \mathcal{T} - (n-1)^{\frac{n-1}{2}} \Delta/\mathcal{T}^{\frac{n-1}{2}}$.

It can be seen that a lower bound for the determinant and an upper bound on the trace are required to get the uniform bounds of the inertia matrix. The following two subsections will address these issues.

### 5.2. A Lower Bound for the Determinant

**Proposition 5.2:** If a positive semi-definite (PSD) matrix $A$ is decomposable as $A = B + C$ such that

1. $B$ and $C$ are PSD,
2. the elements $b_{ij}$ of $B$ are such that $b_{ij} = 0 \ \forall i \neq j, j \neq 1$, and
3. $C = \begin{bmatrix} c_{11} & D \\ E & F \end{bmatrix}$ with $c_{11} \in \mathbb{R}, E, D,$ and $F$ are matrices of appropriate dimensions, then, the determinant of $A$, satisfies $\det(A) \geq b_{11} \det(F)$.

**Proof of Proposition 5.2:** We write $A = B + C = \begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c_{11} & D \\ E & F \end{bmatrix}$. Hence, $\det(A) = \det \begin{bmatrix} c_{11} & 0 \\ E & F \end{bmatrix} + \det \begin{bmatrix} c_{11} & D \\ E & F \end{bmatrix} = b_{11} \det(F) + \det(C)$. Since $C$ is PSD, hence $\det(C) \geq 0$, it follows that $\det(A) \geq b_{11} \det(F)$. ■
Lemma 5.2: The determinant $\Delta$ of the inertia matrix $D(q)$ satisfies

$$\Delta \geq \prod_{k=1}^{n} d_{kk}(k) > 0 \tag{42}$$

where $d_{kk}(k)$ is given by (27).

Proof of Lemma 5.2: From the structure of the inertia matrix described in Fact 3.1, note that

1. $D = \sum_{i=1}^{n} D(k)$ where elements of $D(k)$ are denoted by $d_{ij}(k)$,
2. $D(k)$ is PSD $\forall k$, and
3. $d_{ij}(k) = 0 \forall i > k$ or $j > k, \forall k$,

let $B = D(1)$ and $C = \sum_{i=2}^{n} D(k)$. Then $D$ and $B$ satisfy the conditions of Proposition 5.2. Hence, $\det(D) \geq d_{kk}(1)\det(F)$, where $F$ is an $(n-1) \times (n-1)$ matrix obtained from $D$ by eliminating its first column and first row. Assume that elimination of the first row and first column of $D(k)$ results in the matrix $F(k-1)$. Then the following properties hold

1. $F = \sum_{j=1}^{n-1} F(k)$,
2. $F(k)$ is PSD $\forall k$ (since a principle minor of a PSD matrix is PSD), and
3. the elements of $F(k), f_{ij}(k)$ satisfy $f_{ij}(k) = 0$ $\forall i > k$ or $j > k, \forall k$.

By comparing the properties of $F$ with those of $D$ described earlier, it is clear that an iteration can be set up. Since now $f_{ij}(1) = d_{ij}(2), it is clear that each iteration will “pick up” the diagonal term $d_{kk}(k)$. Hence $\Delta \geq \prod_{k=1}^{n} d_{kk}(k)$. 

The advantage of this proof is that it gives an explicit uniform bound for the determinant as expressed by (42). Since this bound is expressed in terms of inherent link masses and inertias which cannot be arbitrarily close to zero, it follows that the inertia matrix cannot be arbitrarily close to singularity. However, note that (42) does not imply in general that the minimum eigenvalue of the inertia matrix is uniformly bounded. On the other hand, if all the eigenvalues of the inertia matrix are bounded from above, which occurs when the elements of the inertia matrix are bounded, then the minimum eigenvalue of the inertia matrix is indeed uniformly bounded.

5.3. An Upper Bound for the Trace

Lemma 5.3: An upper bound of the trace of $D(q), \overline{\mathcal{F}}$, can be computed using

$$\overline{\mathcal{F}} = \sum_{k=1}^{n} \overline{d_{kk}} \tag{43}$$

where $\overline{d_{kk}}$ is an upper bound of the diagonal elements of $D(q)$ with

$$\overline{d_{kk}} = \left\{ \begin{array}{ll} \sum_{i=k}^{n} m_i \left( \sqrt{l_i^2 + d_i^2} + \sqrt{l_{j-k}^2 + d_j^2} \right)^2 & \text{if joint } k \text{ is prismatic} \\
\lambda_{\max}[I_i] + \lambda_{\max}[I_j] & \text{if joint } k \text{ is revolute} \\
\end{array} \right. \tag{44}$$

$\lambda_{\max}[I_i]$ is the maximum eigenvalue of $I_i$, and

$$\beta_j = \left\{ \begin{array}{ll} 0 & \text{if joint } j \text{ is prismatic} \\
1 & \text{if joint } j \text{ is revolute} \\
\end{array} \right. \tag{45}$$

Proof of Lemma 5.3: The diagonal elements $d_{kk}$ of $D(q)$ are given by

$$d_{kk} = \sum_{i=k}^{n} m_i \left( R_i^T J_{v_i}^k \right)^2 + \left( T_{w_i}^k \right)^2 R_i I_i R_i^T \tag{46}$$

If joint $k$ is prismatic, then (46) becomes

$$d_{kk} = \sum_{i=k}^{n} m_i z_k^T z_{k-1} = \sum_{i=k}^{n} m_i \tag{47}$$

If joint $k$ is revolute then (46) becomes

$$d_{kk} = \sum_{i=k}^{n} m_i \left\| z_{k-1} \times (o_{c_i} - o_{k-1}) \right\|^2 + z_k^T R_i I_i R_i^T z_{k-1} \tag{48}$$

We first compute

$$\left\| z_{k-1} \times (o_{c_i} - o_{k-1}) \right\|^2 = \left\| z_{k-1} \times (o_{c_i} - o_{j-1} + \sum_{j=k}^{i-1} (o_j - o_{j-1})) \right\|^2$$

$$= \left\| z_{k-1} \times (o_{c_i} - o_{j-1} + \sum_{j=k}^{i-1} z_{k-1} \times (o_j - o_{j-1})) \right\|^2$$

$$\leq \left( \left\| z_{k-1} \times (o_{c_i} - o_{j-1}) \right\| + \sum_{j=k}^{i-1} \left\| z_{k-1} \times (o_j - o_{j-1}) \right\|^2 \right)^2 \tag{49}$$
We first consider in (49) the term
\[
\|z_{k-1} \times (o_{ki} - o_{j-1})\| = \|z_{k-1} \times R_{i-1} d_{j_{i-1}}\|
\]
\[
\leq \|z_{k-1}\| \|R_{i-1} d_{j_{i-1}}\|
\]
\[
\leq \sqrt{l_i^2 + d_j^2}
\]
(50)

We now examine in (49) the summation term and consider two cases. If joint \( j \) is revolute, then similarly to the derivation which led to (50), we conclude that
\[
\|z_{k-1} \times (o_j - o_{j-1})\| \leq \sqrt{d_j^2 + d_j^2}
\]
(51)

On the other hand, if joint \( j \) is prismatic then
\[
\|z_{k-1} \times (o_j - o_{j-1})\| = \|z_{k-1} \times R_{j-1} d_{j_{j-1}}\|
\]
\[
= \|z_{k-1} \times R_{j-1}(d_j z_0 + a_j R_{j-1} x_0)\|
\]
\[
= d_j \|z_{k-1} \times z_{j-1} + a_j(z_{k-1} \times x_j)\| = a_j \|z_{k-1} \times x_j\|
\]
(52)

Equality (53) follows from (52) by noting the following two facts. First, \( z_{k-1} \) and \( z_{j-1} \) are parallel vectors as we are considering robots with bounded inertia matrix. Second, in the Denavit–Hartenberg representation, \(|x_j \times z_j| = |x_j \times z_{j-1}| = 1\). Since \( z_{k-1} \) and \( z_{j-1} \) are parallel, it follows that \( \|z_{k-1} \times x_j\| = 1\).

We now use (50), (51), and (53) in equation (49) and conclude
\[
\|z_{k-1} \times (o_{ki} - o_{j-1})\|^2
\]
\[
\leq \left( \sqrt{l_i^2 + d_j^2} + \sum_{j=k}^{i-1} \sqrt{a_j^2 + \beta_j d_j^2} \right)^2
\]
(54)

where \( \beta_j = 1 \) if joint \( j \) is revolute and \( \beta_j = 0 \) if joint \( j \) is prismatic. Finally, we note that as \( z_{k-1} \) and \( R_i \) are of unit norm,
\[
z_{k-1}^T R_i R_i^T z_{k-1} \leq \lambda_{\text{max}}[I_j]
\]
(55)

Expression (43) follows by using (54), (55), (48), and (47).

**Remark 5.2:** The upper bounds of the diagonal elements of \( D(q) \), namely, \( d_{kk} \) given by (44), have the following physical meaning: If joint \( k \) is prismatic, then \( d_{kk} = d_{kk} \) is the sum of the mass of joint \( k \) and of all the links that joint \( k \) is carrying. If joint \( k \) is revolute, then \( d_{kk} \) is the sum of the maximum inertia of link \( k \) and of all the links that joint \( k \) is carrying with all inertias expressed with respect to the axis of rotation of link \( k \). Each maximum inertia is given by the sum of two terms. The first term is the maximum possible inertia of the link with respect to its own axis of rotation. The second term is the product of the link’s mass with the maximum possible distance from the center of mass of the link and the axis of rotation of link \( k \).

### 5.4. Uniform Bounds of the Inertia Matrix

**Theorem 5.1:** For the class of robot manipulators with bounded \( n \times n \) inertia matrix \( D(q) \), the min-bound and the max-bound of the inertia matrix are bounded by the uniform bounds \( \sigma_1 \) and \( \sigma_2 \) as follows
\[
\|D(q)\|_{\text{min}} > \sigma_1 = \frac{(n-1)^{(n-1)} \prod_{k=1}^{n} d_{kk}(k)}{\mathcal{F}^{(n-1)}}
\]
(56)
\[
\|D(q)\|_{\text{max}} < \sigma_2 = \mathcal{F} - \frac{(n-1)^{(n-1)} \prod_{k=1}^{n} d_{kk}(k)}{\mathcal{F}^{(n-1)}}
\]
(57)

where \( d_{kk}(k) \) is given by (27), and \( \mathcal{F} \) by (43).

**Proof of Theorem 5.1:** The positive definite matrix \( D(q) \) clearly satisfies the min-bound and max-bound inequalities of Lemma 5.1. Inequalities (56) and (57) follow immediately since the determinant of \( D(q) \), \( \Delta \), satisfies \( \Delta \geq \prod_{k=1}^{n} d_{kk}(k) \) and \( \mathcal{F} \) is an upper bound of the trace of \( D(q) \).

### 6. AN EXAMPLE

In this section we give a simple example to illustrate the ideas presented in this article. We choose a manipulator with mixed joint configuration which consists of a prismatic joint followed by a revolute joint. This manipulator has therefore a bounded inertia matrix according to the results of section 4. We use the results of Theorem 5.1 to compute the uniform bounds of the inertia matrix.
The two link manipulator is shown in Figure 2. The kinematics and dynamics link parameters are given in Table I. The inertia matrix is given by

\[
\mathbf{D}(\mathbf{q}) = \mathbf{D}(1) + \mathbf{D}(2) = \begin{bmatrix}
  m_1 & 0 \\
  0 & 0
\end{bmatrix} + \begin{bmatrix}
  m_2 & m_2 l_2 c_2 \\
  m_2 l_2 c_2 & m_2 l_2^2 + I_{2z}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  m_1 + m_2 & m_2 l_2 c_2 \\
  m_2 l_2 c_2 & m_2 l_2^2 + I_{2z}
\end{bmatrix}
\]

Therefore, \(d_{11} = m_1 + m_2, \ d_{22} = m_2 l_2^2 + I_{2z}, \ d_{11}(1) = m_1, \ d_{22}(2) = m_2 l_2^2 + I_{2z}\). Using (44), we get \(d_{11} = m_1 + m_2, \ d_{22} = m_2 l_2^2 + I_{2z}\). The uniform bounds \(\sigma_1\) and \(\sigma_2\) of \(\mathbf{D}(\mathbf{q})\) are therefore given by

\[
\sigma_1 = \frac{d_{11}(1)d_{22}(2)}{d_{11} + d_{22}} = \frac{m_1(m_2 l_2^2 + I_{2z})}{m_1 + m_2(1 + l_2^2) + I_{2z}}
\]

(58)

Figure 3 shows plots of the exact values of the min-bound \(\|\mathbf{D}(\mathbf{q})\|_{\text{min}}\) and the max-bound \(\|\mathbf{D}(\mathbf{q})\|_{\text{max}}\), which represent the minimum and maximum eigenvalues of the inertia matrix, respectively, as a function of the link variable \(q_2\). The constant values of the uniform bounds \(\sigma_1\) and \(\sigma_2\) computed in (58) and (59) are also plotted in Figure 3 (horizontal lines). The following numerical values were used to generate the data in Figure 3: \(m_1 = m_2 = 1, \ I_2 = I_{2z} = 0.75, \ l_{c_2} = 1\).

7. CONCLUSIONS

In this article we completely characterized the class of robot manipulators with bounded inertia matrix. We showed that this class includes manipulators with trivial joint configurations (such as all joints

Table I. Kinematics and dynamics link parameters.

<table>
<thead>
<tr>
<th>Link i</th>
<th>Type</th>
<th>(a_i)</th>
<th>(\alpha_i)</th>
<th>(d_i)</th>
<th>(\theta_i)</th>
<th>(q_i)</th>
<th>(m_i)</th>
<th>(l_{c_i})</th>
<th>(I_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>prismatic</td>
<td>0</td>
<td>(\pi/2)</td>
<td>(d_1)</td>
<td>0</td>
<td>(d_1)</td>
<td>(m_i)</td>
<td>(d_1 - d_0)</td>
<td>diag[(l_{1x}) (l_{1y}) (l_{1z})]</td>
</tr>
<tr>
<td>2</td>
<td>revolute</td>
<td>(a_2)</td>
<td>0</td>
<td>(0)</td>
<td>(\theta_2)</td>
<td>(\theta_2)</td>
<td>(m_2)</td>
<td>(l_{c_2})</td>
<td>diag[(l_{2x}) (l_{2y}) (l_{2z})]</td>
</tr>
</tbody>
</table>
are revolute and all joints are prismatic), as well as mixed joint configurations. For this class of manipulators, we gave easily computable uniform bounds for the minimum and maximum eigenvalues of the inertia matrix.

REFERENCES